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Quasi-stationary perturbations of the KdV soliton

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Abstract
In this paper we show how the results of quasi-stationary perturbation theory relate to those based upon squared eigenfunction expansions. We show that the two give the same results for a quasi-stationary approximation and also show how standard adiabatic perturbation theory connects to the quasi-stationary theory.

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1. Introduction

Over the past several decades there has been a growth in the study of integrable and near integrable systems. One of the first generic integrable nonlinear partial differential equations was the Korteweg–deVries equation (KdV), which was derived in 1895 [20] as the equation governing small amplitude water waves exhibiting quadratic nonlinearity and third-order dispersion. Around the sixties, this equation once again arose in other areas and it was shown to be integrable, possessing soliton solutions and an infinite number of conservation laws [2, 25]. A whole industry was born in which over 100 other integrable systems were found in 1D and higher spatial dimensions. Also, a variety of methods have been extended for studying integrability, such as symmetry, Bäcklund and Painlevé methods.

However, it was not long before people were interested in the robustness of solitons. Most of these integrable systems were derived as approximate models of processes in the real world. What if the assumptions were relaxed? Would the soliton survive? How long would it survive? For the KdV equation, the integrable system used in this paper, some of the first perturbation results appeared in Ott and Sudan's 1969 paper [26]. Karpman and Maslov [10–12] and Kaup and Newell [15] presented results based upon the inverse scattering method. One of the first textbook accounts of soliton perturbation theory was in Lamb's book [21]. Menyuk [24] presented results in 1986 using direct methods for studying Hamiltonian perturbations. Herman wrote several papers on a direct method, based upon squared eigenfunction expansions [4–6]. In more recent years, a few other authors have also addressed direct KdV perturbations using either eigenfunction expansions [29, 30] or Green's function methods [23]. There were several other papers on soliton perturbation methods in the early years of soliton research dealing with other near integrable systems. An exhaustive discussion was provided by Kivshar and Malomed in 1989 [17]. Of special interest is the 1981 paper by Kodama and Ablowitz [19] in which they present a direct method using a quasi-stationary approximation.

While many of the perturbation methods have seen obvious connections, it was not entirely clear to the author that the quasi-stationary results in Kodama and Ablowitz [19] had any direct connection to those of others, such as Herman [6] or Yan and Tang [29]. In this paper we will address this problem and show how these seemingly different approaches are connected. We will state this problem more formally in the next section after setting up the perturbation problem.

2. The perturbation problem

We are interested in finding an approximate solution of the perturbed KdV equation

$$u_T + 6uu_x + u_{xxx} = \epsilon R[u] \tag{1}$$

which is close to the KdV soliton, $u_0(z) = 2\eta^2 \operatorname{sech}^2 z$, $z = \eta(x - \xi)$, where $\xi_T = 4\eta^2$. Note that throughout this paper we use standard variable subscript notation to denote differentiation, e.g., $u_x = \frac{\partial u}{\partial x}$.

The now standard procedure is to use multiple time scales and to introduce an expansion of $u(x, T)$. We will pick a slow time τ and fast time t scale, such that

$$\partial_T = \partial_t + \epsilon\partial_\tau. \tag{2}$$

We expand $u(x, T)$ as

$$u(x, T) = u_0(z, \tau) + \epsilon u_1(z, t, \tau) + \cdots . \tag{3}$$

Inserting these expansions in equation (1), we find that $u_0(z)$ satisfies the KdV equation provided $\xi_t = 4\eta^2$ and $u_1(z, t)$ satisfies the forced, linearized KdV equation

$$u_{1_t} + \eta^3 \hat{L} u_1 = R_1 - 4\eta\eta_\tau\phi_1(z) - 4\eta^3\xi_\tau\phi_2(z) \equiv F(z). \tag{4}$$

Here we have defined $R_1 \equiv R[u_0]$ and

$$\begin{aligned} \phi_1(z) &= (1 - z\tanh z)\operatorname{sech}^2 z \\ \phi_2(z) &= \operatorname{sech}^2 z \tanh z. \end{aligned} \tag{5}$$

The linear operator $\hat{L}$ is given by

$$\hat{L} = \frac{\mathrm{d}^3}{\mathrm{d}z^3} + (12\operatorname{sech}^2 z - 4)\frac{\mathrm{d}}{\mathrm{d}z} - 24\operatorname{sech}^2 z \tanh z. \tag{6}$$

We can now state our problems in more detail. We want to solve equation (4) for u_1 and to determine the slow time behaviour of the soliton parameters $\eta(\tau)$ and $\xi(\tau)$. This will give us the amplitude and velocity correction, respectively, of the perturbed soliton. Our goal is to compare the results from the squared eigenfunction expansion, following Yan and Tang [29], to those of the quasi-stationary method of Kodama and Ablowitz [19].

The main assumption of the quasi-stationary method is that u_1 does not depend explicitly upon the fast time scale, t. Thus, we are interested in studying the solution $u_1 = u_1(z)$ of

$$\eta^3 \hat{L} u_1 = R_1 - 4\eta\eta_\tau\phi_1(z) - 4\eta^3\xi_\tau\phi_2(z). \tag{7}$$

We will apply our methods to the damped KdV equation

$$u_t + 6uu_x + u_{xxx} = -\epsilon\gamma u. \tag{8}$$

We show that the solution obtained by the squared eigenfunction method agrees with the results of Kodama and Ablowitz up to solutions of the homogeneous equation $\hat{L}u_1 = 0$. We will further show how this solution is related to the non-quasistationary problem of adiabatic soliton perturbation theory based upon the direct perturbation method as presented by Yan and Tang [29], which is typical of the direct methods. The damped KdV equation is one of the most used test models and arises, for example, in the study of a solitary surface wave in the ocean as it approaches a beach with bottom topography having a constant nonzero slope [8, 9, 18, 22].

3. Quasi-stationary solution—direct integration

We first recall the solution of the quasi-stationary problem by direct integration, similar to the method in Kodama and Ablowitz [19]. We begin with the linearized KdV problem from equation (7) in the form

$$\eta^3 \left[\frac{\mathrm{d}^3}{\mathrm{d}z^3} + (12\,\mathrm{sech}^2 z - 4)\frac{\mathrm{d}}{\mathrm{d}z} - 24\,\mathrm{sech}^2 z \tanh z \right] u_1 = F(z) \tag{9}$$

where $F(z) = R_1 - 4\eta\eta_\tau\phi_1(z) - 4\eta^3\xi_\tau\phi_2(z)$.

Setting $y = \tanh(z)$, $g(y) = u_1(z)$ and $\tilde{F}(y) = F(z)$, we obtain

$$\eta^3 \mathcal{L} g(y) = \tilde{F}(y) \tag{10}$$

where

$$\mathcal{L} = (1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y} + [12(1-y^2) - 4](1-y^2)\frac{\mathrm{d}}{\mathrm{d}y} - 24(1-y^2)y. \tag{11}$$

This is $\hat{L}$ written in the new variable y.

With a little rearranging, this third-order differential equation can be rewritten as

$$\eta^3(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}\left[(1-y^2)\left[\frac{\mathrm{d}}{\mathrm{d}y}(1-y^2)\frac{\mathrm{d}g}{\mathrm{d}y} + \left(12 - \frac{4}{1-y^2}\right)g\right]\right] = \tilde{F}(y). \tag{12}$$

Dividing by $\eta^3(1-y^2)$ and integrating, one obtains the second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}y}\left((1-y^2)\frac{\mathrm{d}g}{\mathrm{d}y}\right) + \left(12 - \frac{4}{1-y^2}\right)g = \frac{1}{1-y^2}\int \frac{\tilde{F}(y)}{\eta^3(1-y^2)}\,\mathrm{d}y. \tag{13}$$

Following Kodama and Ablowitz, we recognize that the associated Legendre polynomial, $P_3^2(y) = 15y(1-y^2)$, is a solution of the homogeneous equation. Thus, one can solve this second-order problem using the method of variation of parameters. We let $g(y) = A(y)P_3^2(y) = 15y(1-y^2)A(y)$. This yields an equation for $A(y)$:

$$15y(1-y^2)^2\left[\frac{\mathrm{d}^2A}{\mathrm{d}y^2} + \frac{2(1-4y^2)}{y(1-y^2)}\frac{\mathrm{d}A}{\mathrm{d}y}\right] = \mathcal{F}(y) \tag{14}$$

where we have defined

$$\mathcal{F}(y) \equiv \frac{1}{1-y^2}\int \frac{\tilde{F}(y)}{\eta^3(1-y^2)}\,\mathrm{d}y. \tag{15}$$

Using the integrating factor $y^2(1-y^2)^3$, we find

$$15\frac{\mathrm{d}}{\mathrm{d}y}\left[y^2(1-y^2)^3\frac{\mathrm{d}A}{\mathrm{d}y}\right] = y(1-y^2)\mathcal{F}(y). \tag{16}$$

This first-order equation can be solved for $A(y)$:

$$A(y) = \int^y \mathrm{d}x \frac{1}{x^2(1-x^2)^3} \int^x \mathrm{d}w \frac{1}{15} w(1-w^2)\mathcal{F}(w). \tag{17}$$

Inserting $\mathcal{F}(y)$ from equation (15) and noting that $u_1(z) = g(y)$, we obtain the general solution to equation (9) as

$$u_1(z) = \left[\frac{y(1-y^2)}{\eta^3} \int^y \mathrm{d}x \frac{1}{x^2(1-x^2)^3} \int^x \mathrm{d}w\, w \int^w \mathrm{d}s \frac{\tilde{F}(s)}{1-s^2} \right]_{y=\tanh z}. \tag{18}$$

This result will be later applied to the damped KdV equation (8) for which

$$\tilde{F}(y) = -2\eta(1-y^2)\left(\eta\gamma + 2\eta_\tau - \eta_\tau \ln\left(\frac{1+y}{1-y}\right) y + 2\eta^2 \xi_\tau y \right). \tag{19}$$

We will first turn to the direct perturbation method using squared eigenfunction expansions, which has generally only been applied to non-quasistationary problems.

4. The eigenfunction expansion method

As noted in the introduction, there have been several approaches to solving the perturbed KdV equation (1). These methods range from using perturbations of the scattering data in the inverse spectral method (transform) [10–12, 15] to using Green's function methods [16, 23], to doing a direct perturbation using squared eigenfunction expansions [6, 13, 14, 27, 29]. These methods have been shown to be equivalent [23, 29]. We will borrow the method and notation of the recent paper of Yan and Tang [29]. In this section we will review the eigenfunction expansion method and in the next section show how it can be applied to the quasi-stationary equation (7). In this way we can make the connection between the early results of Kodama and Ablowitz [19] and these other perturbation studies, which is the goal of this paper.

In short, we have a linear partial differential equation (4) we wish to solve for $u_1(z, t)$. We obtain a solution to this problem as a sum over the eigenfunctions of the operator $\hat{L}$. These eigenfunctions consist of continuous and bound states and are related to the squares of the eigenfunctions of the so-called Lax pair, which is used in the study of the integrability of nonlinear evolution equations and in the development of the inverse scattering transform (IST) [2, 25]. Some authors have found ways to get around IST [23], but it amounts to the same set of eigenfunctions used in an eigenfunction expansion of the Green's function. In fact, one can even relate this basis to a set of squared associated Legendre functions in the case of the KdV equation, as shown in appendix B.

4.1. Solution of Non-quasistationary problems

We seek the eigenfunctions $\phi(z, k)$ and their adjoints $\psi(z, k)$ that can be used in the perturbation expansion. We recall the linear operator $\hat{L}$ and its adjoint $\hat{L}^\dagger$:

$$\hat{L} = \frac{\mathrm{d}^3}{\mathrm{d}z^3} + (12\,\mathrm{sech}^2 z - 4)\frac{\mathrm{d}}{\mathrm{d}z} - 24\,\mathrm{sech}^2 z \tanh z. \tag{20}$$

$$\hat{L}^\dagger = \frac{\mathrm{d}^3}{\mathrm{d}z^3} + (12\,\mathrm{sech}^2 z - 4)\frac{\mathrm{d}}{\mathrm{d}z}. \tag{21}$$

The eigenfunctions of these operators satisfy the eigenvalue problems

$$\begin{aligned} \hat{L}\phi &= \lambda\phi \qquad && \lambda = -\mathrm{i}k(k^2+4) \\ \hat{L}^\dagger\psi &= \lambda'\psi \qquad && \lambda' = \mathrm{i}k(k^2+4). \end{aligned} \tag{22}$$

These problems have both continuous and bound eigenstates. The continuous states are

$$\phi(z,k) = \frac{1}{\sqrt{2\pi}k(k^2+4)}[k(k^2+4)+4\mathrm{i}(k^2+2)\tanh z - 8k\tanh^2 z - 8i\tanh^3 z]\,\mathrm{e}^{\mathrm{i}kz}$$
$$\psi(z,k) = \frac{1}{\sqrt{2\pi}(k^2+4)}[k^2 - 4\mathrm{i}k\tanh z - 4\tanh^2 z]\,\mathrm{e}^{-\mathrm{i}kz}. \tag{23}$$

The bound, or discrete, states are given by

$$\begin{aligned} \phi_1(z) &= (1 - z\tanh z)\,\mathrm{sech}^2 z \qquad & \phi_2(z) &= \mathrm{sech}^2 z\tanh z \\ \psi_1(z) &= \mathrm{sech}^2 z & \psi_2(z) &= \tanh z + z\,\mathrm{sech}^2 z. \end{aligned} \tag{24}$$

We note that $\hat{L}\phi_1 = -8\phi_2$ and $\hat{L}\phi_2 = 0$.

Yan and Tang [29] provided the needed properties of these eigenfunctions. These states satisfy the orthogonality conditions

$$\begin{aligned} &\int_{-\infty}^{\infty} \phi(z,k)\psi(z,k')\,\mathrm{d}z = \delta(k-k') \\ &\int_{-\infty}^{\infty} \phi_j(z)\psi_\ell(z)\,\mathrm{d}z = \delta_{j,\ell} \qquad j,k = 1,2. \end{aligned} \tag{25}$$

The completeness relation can be written as

$$P\int_{-\infty}^{\infty} \phi(z,k)\psi(z',k)\,\mathrm{d}k + \sum_{j=1}^{2}\phi_j(z)\psi_j(z') = \delta(z-z'). \tag{26}$$

Here P denotes the Cauchy principal value [1, 3] since these integrals not only have infinite limits, but also involve a pole at $k = 0$ on the real axis.

We can now construct the solution $u_1(z,t)$ using this basis. We expand the forcing term in equation (4) as

$$F(z) = P\int_{-\infty}^{\infty} f(k)\phi(z,k)\,\mathrm{d}k + \sum_{j=1}^{2} f_j\phi_j(z). \tag{27}$$

Using the orthogonality conditions, the expansion coefficients are found using

$$\begin{aligned} f(k) &= \int_{-\infty}^{\infty} F(z)\psi(z,k)\,\mathrm{d}z = \int_{-\infty}^{\infty} R[u_0(z)]\psi(z,k)\,\mathrm{d}z \\ f_j &= \int_{-\infty}^{\infty} F(z)\psi_j(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} R[u_0(z)]\psi_j(z)\,\mathrm{d}z \qquad j = 1,2. \end{aligned} \tag{28}$$

Note that in general $F = F(z,t)$ and its expansion coefficients may have an explicit fast time dependence. However, for the example in this paper such a time dependence is not needed.

We can also expand $u_1(z,t)$ in the basis as

$$u_1(z,t) = P\int_{-\infty}^{\infty} U(t,k)\phi(z,k)\,\mathrm{d}k + \sum_{j=1}^{2} U_j(t)\phi_j(z). \tag{29}$$

Inserting this expansion in equation (4), $u_{1_t} + \eta^3\hat{L}u_1 = F$, we find that the coefficients satisfy the equations

$$\begin{aligned} &U_t + \eta^3\lambda(k)U = f(k) \qquad && U(0,k) = 0 \\ &U_{1t} = f_1 && U_1(0) = 0 \\ &U_{2t} - 8\eta^3 U_1 = f_2 && U_2(0) = 0. \end{aligned} \tag{30}$$

In principle, we have solved the forced, linearized KdV equation after solving these simple first-order ordinary differential equations. However, solutions of the U_i equations may lead to solution growth in time, or secularities. This can force the secularity conditions $f_i \equiv 0, i = 1, 2$. These typically will give information about the first-order effects of the perturbation on the soliton amplitude and speed. Next we will show how the assumption of quasistationarity changes the system in (30) .

4.2. Solution of quasi-stationary problem

We are interested in seeing how the solution from direct integration in the last section is related to that obtained from an expansion in the squared eigenfunction basis. We have found that the expansion coefficients in the non-quasistationary problem satisfy a set of ordinary differential equations (30). However, the time derivative is with respect to the fast time scale. For the quasi-stationary problem there is no fast time scale. So, we need to redo the perturbation expansion method with this in mind for equation (7). Proceeding as before, we have

$$u_1(z) = P\int_{-\infty}^{\infty} U(k)\phi(z,k)\,\mathrm{d}k + U_1\phi_1(z) + U_2\phi_2(z). \tag{31}$$

Then

$$\hat{L}u_1 = P\int_{-\infty}^{\infty} U(k)[-\mathrm{i}k(k^2+4)]\phi(z,k)\,\mathrm{d}k - 8U_1\phi_2(z). \tag{32}$$

Expanding $F(z)$, as before,

$$F(z) = P\int_{-\infty}^{\infty} f(k)\phi(z,k)\,\mathrm{d}k + f_1\phi_1(z) + f_2\phi_2(z) \tag{33}$$

we can equate the coefficients to obtain

$$U(k)\eta^3[-\mathrm{i}k(k^2+4)] = f(k) = \int_{-\infty}^{\infty} F(z)\psi(z,k)\,\mathrm{d}z \tag{34}$$

$$0 = f_1 = \int_{-\infty}^{\infty} F(z)\,\mathrm{sech}^2\,z\,\mathrm{d}z \tag{35}$$

$$-8\eta^3 U_1 = f_2 = \int_{-\infty}^{\infty} F(z)[\tanh z + z\,\mathrm{sech}^2\,z]\,\mathrm{d}z. \tag{36}$$

In this case, there is no growth in time. However, we see that there is a solvability condition, $f_1 \equiv 0$ from (34). Thus, we do recover one of our previous secularity conditions automatically. Of course, this is not unexpected, since $v = \mathrm{sech}^2\,z$ is a solution of the adjoint problem $\hat{L}^\dagger v = 0$, where $\hat{L}^\dagger$ was defined in (21).

This completes the general solution of the quasi-stationary problem using the eigenfunction expansion method. We are now ready to apply our solution to a particular perturbation and then compare this solution to that obtained directly in section 3.

5. Application to the damped KdV equation

In this section we will apply the previous general results to the damped KdV equation (8) in order to see the connection between the two methods considered in this paper. We start by comparing the quasi-stationary results from both methods. We will then compare the quasi-stationary result with the standard result from adiabatic perturbation theory.

5.1. Quasi-stationary solution: direct method

We first evaluate the solution in equation (18), using $\tilde{F}(z)$ from equation (19). We note that the proper solution of the adjoint problem, $\hat{L}u = 0$, is $u_0 = 2\eta^2 \operatorname{sech}^2 z$. Thus, the compatibility condition that we have seen in earlier sections is given by

$$\int_{-\infty}^{\infty} F(z) \operatorname{sech}^2 z \, \mathrm{d}z = 0 \tag{37}$$

where $F(z) = R[u_0] - 4\eta\eta_\tau\phi_1(z) - 4\eta^3\xi_\tau\phi_2(z)$. For the damped KdV equation we set $R[u_0] = -\gamma u_0(z)$. This leads to

$$\eta_\tau = -\tfrac{2}{3}\eta\gamma. \tag{38}$$

So, the forced, linearized problem takes the form

$$\eta^3\hat{L}u_1 = \tfrac{2}{3}\gamma\eta^2 \operatorname{sech}^2 z(1 - 4z\tanh z) - 4\eta^3\xi_\tau \operatorname{sech}^2 z \tanh z. \tag{39}$$

Using this forcing term in the general solution (18), we find that

$$u_1(z) = \frac{\gamma}{6\eta}\left[-1 + \tanh z + 3\left(1 + \frac{\eta}{\gamma}\xi_\tau\right)(1 - z\tanh z)\operatorname{sech}^2 z \right.$$
$$\left. + z(2 - z\tanh z)\operatorname{sech}^2 z\right] - \left(\frac{5\gamma}{12\eta} + \frac{1}{2}\xi_\tau\right)\operatorname{sech}^2 z \tanh z. \tag{40}$$

There are extra terms that are proportional to one of the solutions of the homogeneous problem. In particular, the general solution to $\hat{L}v(z) = 0$ is

$$v(z) = c_1\cosh^2 z + c_2\operatorname{sech}^2 z\tanh z + c_3[-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z]. \tag{41}$$

However the first term leads to an unbounded solution for large z, so we can set $c_1 = 0$. Now the above solution can be rewritten in the general form

$$u_1(z) = \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z$$
$$+ C_1\operatorname{sech}^2 z\tanh z + C_2(-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z) \tag{42}$$

where C_1 and C_2 are arbitrary functions independent of z. Inserting this solution into $\eta^3\hat{L}u$ yields

$$\eta^3\hat{L}u_1 = \tfrac{2}{3}\gamma\eta^2\operatorname{sech}^2 z(1 - 4z\tanh z) - 4\eta^3\xi_\tau\operatorname{sech}^2 z\tanh z \tag{43}$$

confirming that this is a solution of the linearized problem in equation (39).

5.2. Quasi-stationary solution: eigenfunction method

In order to compare the previous solution to the eigenfunction expansion solution (29), we need to compute the inner products in (34). We first obtain for the expansion coefficients of $F(z)$

$$f(k) = \frac{\sqrt{2\pi}}{3}\frac{\gamma\eta^2 k}{\sinh\left(\frac{\pi k}{2}\right)} \tag{44}$$

$$f_1 = -\frac{8}{3}\gamma\eta^2 - 4\eta\eta_\tau \tag{45}$$

$$f_2 = -4\eta^3\xi_\tau. \tag{46}$$

For $f_1 = 0$, we get the standard η-dependence as given in equation (38).

The harder part of this computation is the evaluation of the sum over the continuous states:

$$I = P\int_{-\infty}^{\infty} U(k)\phi(z,k)\,\mathrm{d}k. \tag{47}$$

Solving for $U(k)$ in (34) in terms of $f(k)$ in (44), we have

$$U(k) = -\frac{\sqrt{2\pi}}{3\mathrm{i}\eta\gamma}\frac{1}{(k^2+4)\sinh\left(\frac{\pi k}{2}\right)}. \tag{48}$$

Combining this expression with $\phi(z,k)$ in equation (23) yields the integral

$$I = \frac{\mathrm{i}\gamma}{3\eta}P\int_{-\infty}^{\infty}\frac{k(k^2+4)+4\mathrm{i}(k^2+2)\tanh z - 8k\tanh^2 z - 8\mathrm{i}\tanh^3 z}{k(k^2+4)^2\sinh\left(\frac{\pi k}{2}\right)}\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k. \tag{49}$$

The computation of this contour integral is treated in appendix A. The result is

$$I = \frac{\gamma}{6\eta}\left[-\left(\frac{\pi^2}{12}+z^2+\frac{3}{2}\right)\operatorname{sech}^2 z\tanh z + \tanh z + 2z\operatorname{sech}^2 z\right]. \tag{50}$$

The full solution of the quasi-stationary problem using the eigenfunction expansion method can now be found. We insert (50) and $U_2 = 0$ into the solution (31) and absorb any terms proportional to $\operatorname{sech}^2 z\tanh z$. This gives the solution

$$u_1 = \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z + \tilde{C}\operatorname{sech}^2 z\tanh z. \tag{51}$$

We note that insertion into the linear operator $\eta^3\hat{L}u$ yields

$$\eta^3\hat{L}u_1 = \tfrac{2}{3}\gamma\eta^2\operatorname{sech}^2 z(1-4z\tanh z) - 4\eta^3\xi_\tau\operatorname{sech}^2 z\tanh z. \tag{52}$$

Therefore, our solution is a solution of the linearized equation (39).

6. Results and discussion

We have considered the perturbation of the KdV equation under the assumption of quasi-stationarity. For the damped KdV equation, this amounts to solving the equation

$$\eta^3\hat{L}u_1 = \tfrac{2}{3}\gamma\eta^2\operatorname{sech}^2 z(1-4z\tanh z) - 4\eta^3\xi_\tau\operatorname{sech}^2 z\tanh z \tag{53}$$

where

$$\hat{L} = \frac{\mathrm{d}^3}{\mathrm{d}z^3} + (12\operatorname{sech}^2 z - 4)\frac{\mathrm{d}}{\mathrm{d}z} - 24\operatorname{sech}^2 z\tanh z. \tag{54}$$

Using direct integration methods, we obtained (42),

$$u_1(z) = \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z$$
$$+ C_1\operatorname{sech}^2 z\tanh z + C_2(-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z). \tag{55}$$

Using an eigenfunction expansion method, we obtained (51),

$$u_1 = \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z + \tilde{C}\operatorname{sech}^2 z\tanh z. \tag{56}$$

In both cases we have found that the same compatibility condition is needed, which leads to the time dependence $\eta_\tau = -\frac{2}{3}\eta\gamma$.

However, we see that these solutions differ by a solution of the homogeneous equation, $v(z) \equiv -1 + 3(1 - z\tanh z)\operatorname{sech}^2 z$. This term did not appear in the general theory for the eigenfunction expansion method. Since the homogeneous problem is a third-order ordinary differential equation, one would expect another arbitrary term. In fact, it can be shown that this missing term arises from the $k = 0$ contribution to the integral over the continuous states.

We can write $v(z) = -1 + 3(1 - z\tanh z)\operatorname{sech}^2 z = -1 + 3\phi_1(z)$. We seek an expansion of $v(z)$ in terms of the basis to see where it arises in the theory. Using our basis of eigenfunctions, we find that

$$1 = P\int_{-\infty}^{\infty} h(k)\phi(z,k)\,\mathrm{d}k + 2\phi_1(z) \tag{57}$$

where $h(k) = \frac{k^2-4}{k^2+4}\frac{\delta(k)}{\sqrt{\pi}}$. Thus, $v(z)$ has the expansion

$$v(z) = -P\int_{-\infty}^{\infty} h(k)\phi(z,k)\,\mathrm{d}k + \phi_1(z). \tag{58}$$

A computation of the integral yields the correct form for $v(z)$. Therefore, the $k = 0$ pole contributes to this particular solution of the homogeneous problem, $\hat{L}v = 0$, accounting for the missing term.

We now compare our general solution to that of Kodama and Ablowitz [19]. They gave the solution for $u_1(z)$ as

$$u_1(z) = \frac{\gamma}{6\eta}\left[-1 + \tanh z + 3\left(1 + \frac{\eta}{\gamma}\xi_\tau\right)(1 - z\tanh z)\operatorname{sech}^2 z + z(2 - z\tanh z)\operatorname{sech}^2 z\right]. \tag{59}$$

This agrees with the first solution (42) up to terms proportional to $\operatorname{sech}^2 z\tanh z$. In fact, this $u_1(z)$ can be rewritten as

$$\begin{aligned} u_1(z) = {} & \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z \\ & + \frac{\gamma}{6\eta}(-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z) \end{aligned} \tag{60}$$

showing that the extra term is proportional to $v(z)$ above.

This leads to asking how one can absorb the arbitrary terms in the general solution (42). We can require that $u_1(z)$ approaches 0 for large z. This yields $C_2 = \frac{\gamma}{6\eta}$. Inserting this value into (42) gives the same factor in Kodama and Ablowitz's solution as given by equation (60). Higher order derivatives automatically vanish asymptotically.

One can absorb the C_1 term into the leading order solution by picking the correct phase constant when solving for the slow time dependence for ξ, which still needs to be handled. In non-quasistationary perturbation theory this is done through the secularity conditions.

So far we have one common secularity, or compatibility, condition. In both problems we arrived at the condition

$$\int_{-\infty}^{\infty} F(z)\operatorname{sech}^2 z\,\mathrm{d}z = 0. \tag{61}$$

However, in the non-quasistationary perturbation theory, there is a second condition, which gives the correction to the soliton velocity [4]

$$\int_{-\infty}^{\infty} F(z)[\tanh z + z\operatorname{sech}^2 z + \tanh^2 z]\,\mathrm{d}z = 0. \tag{62}$$

This is considered in the quasi-stationary problem by Kodama and Ablowitz [19] through the use of conservation of energy. Namely, from the damped KdV equation, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x = -2\epsilon\gamma\int_{-\infty}^{\infty} u^2\,\mathrm{d}x. \tag{63}$$

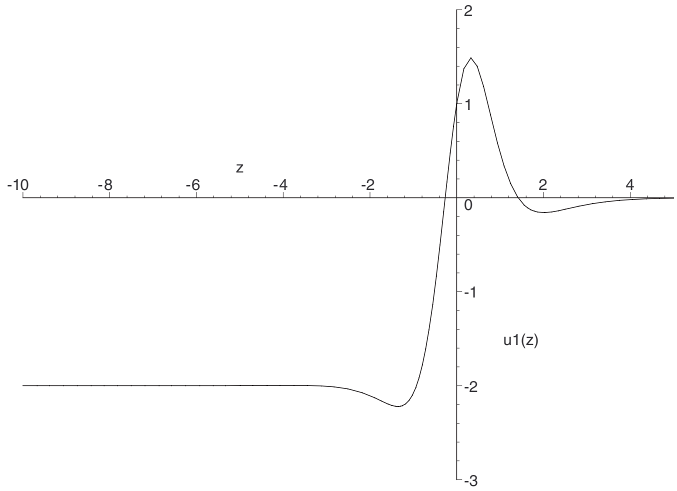

Figure 1. This is a scaled plot of the first-order solution $u_1(z)$ in equation (66).

Using this relation, Kodama and Ablowitz show that $\xi_\tau = -\frac{\gamma}{3\eta}$. This result agrees with non-quasistationary perturbation theory. Similar results based upon conservation laws were found by Herman [7]. Inserting this result in all three solutions leads to the same results for $u_1(z)$.

One still has to account for the fact that there is a remaining arbitrariness in the solution in the form of the unresolved $\operatorname{sech}^2 z \tanh z$ terms. Note that to order ϵ we have

$$\begin{aligned} u_0(z) &= 2\eta^2 \operatorname{sech}^2(\eta(x - \xi + \epsilon x_0)) \\ &\approx 2\eta^2 \operatorname{sech}^2 z - 4\epsilon x_0 \eta^3 \operatorname{sech}^2 z \tanh z. \end{aligned} \tag{64}$$

Integrating $\xi_\tau = -\frac{\gamma}{3\eta}$ while using $\eta_\tau = -\frac{2}{3}\eta\gamma$, we have

$$\xi = \xi_0 - \tfrac{1}{2}\,\mathrm{e}^{2\gamma\tau/3}. \tag{65}$$

Thus, we can pick $C_2 = -4x_0\eta^3$ in order to adjust the phase of the perturbed solution at $\tau = 0$.

In summary, we can write the solution of the forced, linearized KdV equation under the quasi-stationary assumption as

$$u_1(z) = \frac{\gamma}{6\eta}[-1 + \tanh z + 2(1 - z\tanh z)\operatorname{sech}^2 z + z(2 - z\tanh z)\operatorname{sech}^2 z]. \tag{66}$$

This form has resulted from using both direct and eigenfunction expansion methods. For completeness, we can plot this solution up to the amplitude factor of $\frac{\gamma}{6\eta}$ as a function of z. In figure 1 we plot $u_1(z)$. To see how it affects the full perturbed solution, we plot $u(z) = u_0(z) + \epsilon u_1(z)$ for $\epsilon = 0.01$ and $\gamma = 6\eta$ in figure 2. We note that Kodama and Ablowitz [19] give the range of validity of u_1 as $|z| \ll \epsilon^{-1/2}$. For $\epsilon = 0.01$ this would give $|z| \ll 10$. In figure 3 we zoom in to see the developing shelf behind the perturbed soliton.

Finally, we should compare the quasi-stationary results to the solution to the full non-quasistationary problem. This is given in [29] as (using the secularity conditions to eliminate any bound state terms!)

$$u_1(z,t) = \frac{\mathrm{i}\gamma}{3\eta} P \int_{-\infty}^{\infty} \frac{1 - \mathrm{e}^{\mathrm{i}k(k^2+4)\eta^3 t}}{k(k^2+4)^2 \sinh\left(\frac{\pi k}{2}\right)}\, p(z,k)\, \mathrm{e}^{\mathrm{i}kz}\, \mathrm{d}k \tag{67}$$

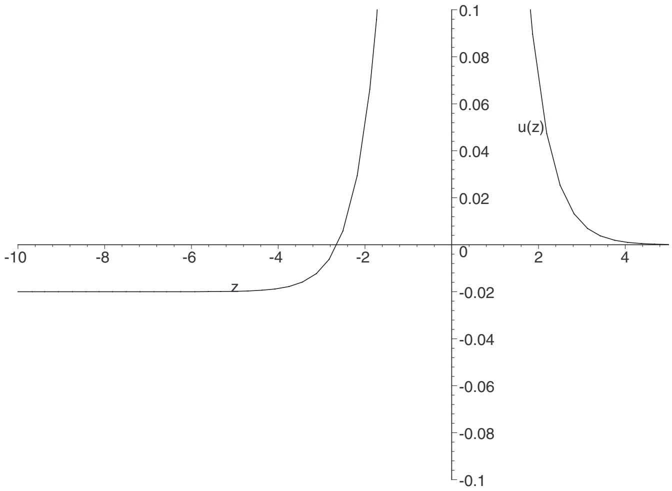

Figure 2. In this figure we plot $u(z) = u_0(z) + \epsilon u_1(z)$ for $\epsilon = 0.01$ and $\gamma = 6\eta$.

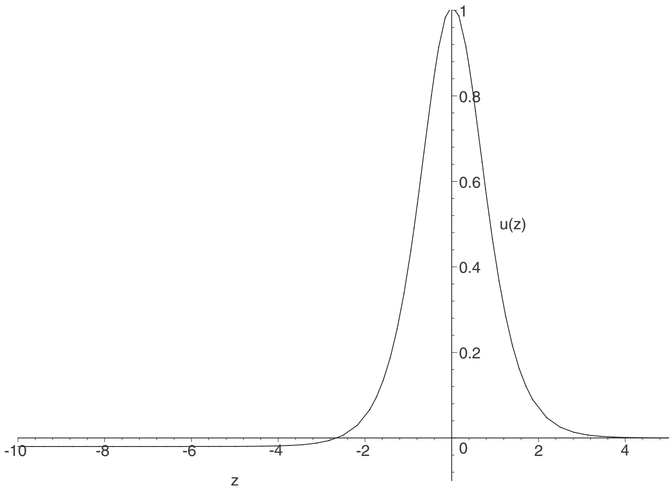

Figure 3. In this figure we zoom into the previous plot of $u(z) = u_0(z) + \epsilon u_1(z)$ for $\epsilon = 0.01$ and $\gamma = 6\eta$ noting the shelf behind the perturbed soliton.

where

$$p(z, k) = k(k^2 + 4) + 4\mathrm{i}(k^2 + 2)\tanh z - 8k\tanh^2 z - 8\mathrm{i}\tanh^3 z. \tag{68}$$

This solution is also obtained from the analysis in section 4 by solving for $U(k, t)$.

We note that the difference between this solution and the quasi-stationary solution (42) is the absence of the explicit time dependence given by the exponential term. By replacing t by t/ϵ and letting ϵ get small, this term leads to a rapid oscillation and leaves only the quasi-stationary solution. An interpretation of the quasi-stationary approximation is that the solution represents a long time t/ϵ behaviour when the time changes are a small correction to the solution in a comoving system as represented by our use of $z \approx \eta(x - 4\eta^2 t)$. Moreover, the time-dependent term in this solution accounts for the dispersive radiative terms, which are not accounted for in the quasi-stationary solution. Kodama and Ablowitz show how these can be obtained by solving a linearized KdV equation outside the region of validity for the quasi-stationary solution and using matched asymptotic expansions.

7. Conclusion

We have compared several methods for solving the perturbed KdV equation. The goal of this paper was to connect the direct integration of the quasi-stationary perturbation presented by Kodama and Ablowitz [19] to the now standard direct method using eigenfunction expansions [29]. We have found that these methods typically agree up to solutions of the homogeneous linearized KdV equation under the assumption of quasi-stationarity. Using appropriate boundary conditions and adjusting the initial phase, we have determined that the first-order correction agrees with that of Kodama and Ablowitz:

$$u_1(z) = \frac{\gamma}{6\eta}[-1 + \tanh z + 2(1 - z\tanh z)\,\mathrm{sech}^2 z + z(2 - z\tanh z)\,\mathrm{sech}^2 z]. \tag{69}$$

As it was not in our interest to explore these solutions in detail, we have not discussed some of the known results, such as the shape of the shelf that develops behind the perturbed soliton and the range of validity of the linearized solution, $|z| \ll \epsilon^{-1/2}$. Kodama and Ablowitz use matched asymptotic expansions in order to obtain a uniform solution outside the region of validity of the quasi-stationary solution in order to ascertain the shelf and emitted linear dispersive waves. Yan and Tang look at the large z behaviour and obtain similar results by studying the asymptotics of the integral over the continuous states.

Appendix A. Evaluation of integral (49)

In this appendix we discuss the computation of the integral

$$I = \frac{\mathrm{i}\gamma}{3\eta} P\int_{-\infty}^{\infty} \frac{k(k^2+4) + 4\mathrm{i}(k^2+2)\tanh z - 8k\tanh^2 z - 8\mathrm{i}\tanh^3 z}{k(k^2+4)^2\sinh\left(\frac{\pi k}{2}\right)}\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k. \tag{A.1}$$

This can be carried out by considering the evaluation of the following integrals:

$$I_1 = P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kz}}{(k^2+4)\sinh\left(\frac{\pi k}{2}\right)}\,\mathrm{d}k. \tag{A.2}$$

$$I_2 = P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kz}}{k(k^2+4)\sinh\left(\frac{\pi k}{2}\right)}\,\mathrm{d}k. \tag{A.3}$$

$$I_3 = P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kz}}{(k^2+4)^2\sinh\left(\frac{\pi k}{2}\right)}\,\mathrm{d}k. \tag{A.4}$$

$$I_4 = P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kz}}{k(k^2+4)^2\sinh\left(\frac{\pi k}{2}\right)}\,\mathrm{d}k. \tag{A.5}$$

Then the full solution would be given by

$$I = \frac{\mathrm{i}\gamma}{3\eta}[I_1 + 4\mathrm{i}(I_2 - 2I_4)\tanh z - 8I_3\tanh^2 z - 8\mathrm{i}I_4\tanh^3 z] \tag{A.6}$$

where each integral can be evaluated using complex contour integral methods.

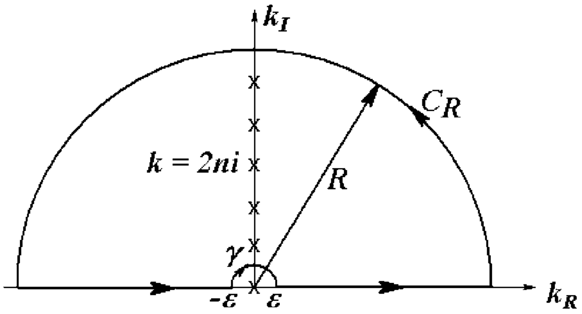

Figure 4. This is the contour needed to perform the Cauchy principal value integrals for $z > 0$ [1, 3]. A similar contour in the lower half complex plane can be used for $z < 0$.

For $z > 0$, we consider the contour given in figure 4. We close the contour in the upper half plane and go around the $k = 0$ pole with a semicircle of radius ϵ and then let ϵ approach zero. The form of the integration result is given by ($m = 1, 2, 3, 4$)

$$I_m = 2\pi \mathrm{i} \sum_{n=1}^{\infty} \mathrm{Res}[f_m(k_n); k_n = 2\mathrm{i}n] + \pi \mathrm{i}\, \mathrm{Res}[f_m(k); k = 0]. \tag{A.7}$$

For the above integrals, we identify the f_m as

$$f_1 = \frac{\mathrm{e}^{\mathrm{i}kz}}{(k^2+4)\sinh\left(\frac{\pi k}{2}\right)} \tag{A.8}$$

$$f_2 = \frac{\mathrm{e}^{\mathrm{i}kz}}{k(k^2+4)\sinh\left(\frac{\pi k}{2}\right)} \tag{A.9}$$

$$f_3 = \frac{\mathrm{e}^{\mathrm{i}kz}}{(k^2+4)^2\sinh\left(\frac{\pi k}{2}\right)} \tag{A.10}$$

$$f_4 = \frac{\mathrm{e}^{\mathrm{i}kz}}{k(k^2+4)^2\sinh\left(\frac{\pi k}{2}\right)} \tag{A.11}$$

and we find that

$$I_1 = \frac{\mathrm{i}}{2} - \frac{\mathrm{i}}{4}(4z+1)\,\mathrm{e}^{-2z} - \mathrm{i}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\,\mathrm{e}^{-2nz} \tag{A.12}$$

$$I_2 = -\frac{z}{2} - \frac{1}{8}(4z+3)\,\mathrm{e}^{-2z} - \frac{1}{4}\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n^2-1)}\,\mathrm{e}^{-2nz} \tag{A.13}$$

$$I_3 = \frac{\mathrm{i}}{8} - \frac{\mathrm{i}}{192}(24z^2+24z+2\pi^2+9)\,\mathrm{e}^{-2z} + \frac{\mathrm{i}}{4}\sum_{n=2}^{\infty}\frac{(-1)^n}{(n^2-1)^2}\,\mathrm{e}^{-2nz} \tag{A.14}$$

$$I_4 = -\frac{z}{8} - \frac{1}{384}(24z^2+48z+2\pi^2+33)\,\mathrm{e}^{-2z} + \frac{1}{8}\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n^2-1)^2}\,\mathrm{e}^{-2nz}. \tag{A.15}$$

Each infinite series can be summed in terms of known functions using partial fraction decomposition and the series summations [28]

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, \mathrm{e}^{-2nz}}{n} = \ln(1+\mathrm{e}^{-2z}) \tag{A.16}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, \mathrm{e}^{-2nz}}{n^2} = \mathrm{Li}_2(\mathrm{e}^{-2z}) \equiv \int_0^{\mathrm{e}^{-2z}} \frac{\ln(1+x)}{x} \, \mathrm{d}x. \tag{A.17}$$

Inserting the computed integrals into equation (A.6) we obtain the result shown in equation (50).

For $z < 0$, the above integrals can be computed by following a contour in the lower half complex k-plane and going around the $k = 0$ pole. The results are

$$I_1 = \frac{\mathrm{i}}{2} + \frac{\mathrm{i}}{4}(4z-1)\,\mathrm{e}^{2z} - \mathrm{i}\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1}\,\mathrm{e}^{2nz} \tag{A.18}$$

$$I_2 = -\frac{z}{2} - \frac{1}{8}(4z-3)\,\mathrm{e}^{2z} + \frac{1}{4}\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n^2-1)}\,\mathrm{e}^{2nz} \tag{A.19}$$

$$I_3 = \frac{\mathrm{i}}{8} - \frac{\mathrm{i}}{192}(24z^2 - 24z + 2\pi^2 + 9)\,\mathrm{e}^{2z} + \frac{\mathrm{i}}{4}\sum_{n=2}^{\infty} \frac{(-1)^n}{(n^2-1)^2}\,\mathrm{e}^{2nz} \tag{A.20}$$

$$I_4 = -\frac{z}{8} - \frac{1}{384}(24z^2 - 48z + 2\pi^2 + 33)\,\mathrm{e}^{2z} - \frac{1}{8}\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n^2-1)^2}\,\mathrm{e}^{2nz}. \tag{A.21}$$

Inserting these computed integrals into equation (A.6), we once again obtain the result shown in equation (50).

Appendix B. Relation to squared associated Legendre functions

Several authors have noted that one does not need the full power of the IST to find the eigenfunction basis or Green's functions needed to solve the linearized problem [23]. For the KdV problem, one can show that the basis and its adjoint basis can be written in terms of associated Legendre functions. Letting $P_1^\mu \equiv P_1^\mu(\tanh z)$, we can show that

$$\hat{L}^\dagger \big(P_1^\mu\big)^2 = 8\mu(\mu^2-1)\big(P_1^\mu\big)^2 \tag{B.1}$$

$$\hat{L}\frac{\mathrm{d}}{dz}\big(P_1^\mu\big)^2 = 8\mu(\mu^2-1)\frac{\mathrm{d}}{\mathrm{d}z}\big(P_1^\mu\big)^2. \tag{B.2}$$

In fact, one can write the squared functions in terms of the unsquared ones $\big(P_n^\mu = P_n^\mu(\tanh z)\big)$:

$$\begin{aligned}
\big(P_1^\mu\big)^2 &= \frac{\tanh^2 z - 2\mu\tanh z + \mu^2}{4\Gamma(2-\mu)^2}\,\mathrm{e}^{2\mu z} = A\left[\frac{1}{6}(\mu+1)P_0^{2\mu} - \frac{1}{3}(2\mu-1)P_2^{2\mu}\right] \\
\frac{\mathrm{d}}{\mathrm{d}z}\big(P_1^\mu\big)^2 &= \frac{\mu(\mu^2-1) + (1-2\mu^2)\tanh z + 2\mu\tanh^2 z - \tanh^3 z}{2\Gamma(2-\mu)^2}\,\mathrm{e}^{2\mu z} \\
&= A\left[\frac{1}{3}\mu(\mu+1)P_0^{2\mu} - \frac{2}{5}(2\mu^2+\mu-1)P_1^{2\mu} - \frac{2}{15}(4\mu^2-8\mu+3)P_3^{2\mu}\right]
\end{aligned} \tag{B.3}$$

where

$$A = \frac{\Gamma(1-2\mu)}{\Gamma(1-\mu)\Gamma(2-\mu)}.$$

Finally, the Yan and Tang [29] eigenfunctions (which are related to those in [6]) are related as

$$\sqrt{2\pi}k(4+k^2)\psi(z,k) = -16\Gamma\left(2+\frac{k^2}{4}\right)^2\left[P_1^{-\mathrm{i}k/2}(\tanh z)\right]^2$$

$$\sqrt{2\pi}\mathrm{i}k(4+k^2)\phi(z,k) = -16\Gamma\left(2+\frac{k^2}{4}\right)^2\frac{\mathrm{d}}{\mathrm{d}z}\left[P_1^{\mathrm{i}k/2}(\tanh z)\right]^2.$$

It would be interesting to see how these are related to the quasi-stationary problem, where we had used the method of variation of parameters in which the homogeneous solution is $P_2^3(\tanh z)$. There might also be a connection of associated Legendre functions to perturbation bases for other near integrable systems. These connections are left for a later paper.

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